APPENDIX G
CONTENT IN FOCUS

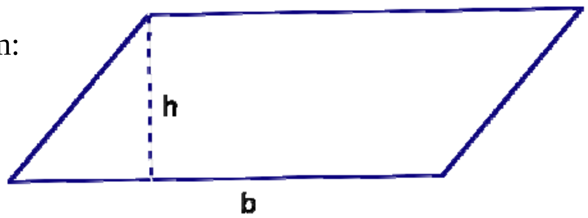
Standard 4 – Geometry and Measurement

GLE 0506.4.1 Use basic formulas and visualization to find the area of geometric figures.

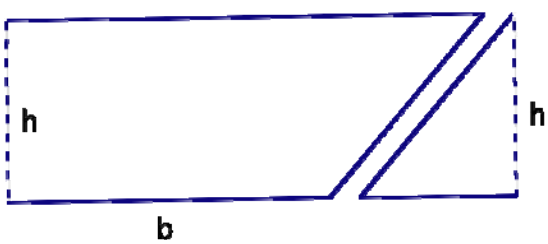
✓ **0506.4.1 Develop the formula for the area of a triangle as it relates to the area of a parallelogram/rectangle.**

Derive the **area formula for a parallelogram** from the area formula for a rectangle: $A = lw$.

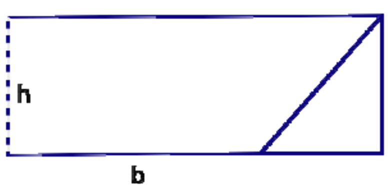
Start with a parallelogram:



Cut off the triangle formed on the left side by the dashed line segment *h* and slide the triangle to the right edge of the figure to form a rectangle.



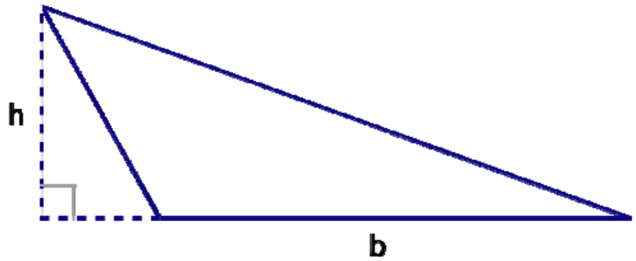
Final rectangle:



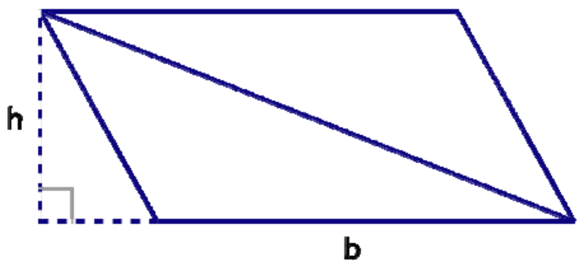
Apply formula for area of a rectangle: $A = lw$ where length is b and height is h to derive area formula for a parallelogram: $A = bh$.

Now students can use *A* = *bh* to derive the **area formula for a triangle**.

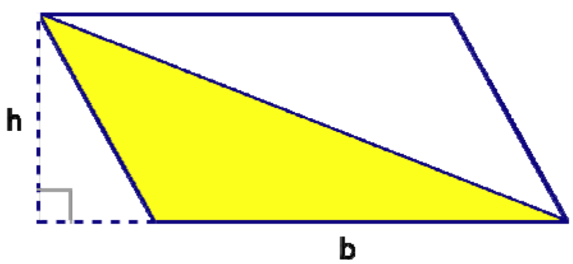
Start with a triangle.



Make a copy of it (for example, on patty paper) and use both triangles to construct a parallelogram.



Notice that the parallelogram has the same base and height as the original triangle. If we found the area of the parllelogram, we would use A = bh, but we want the area of one triangle, so we we need $A = \frac{1}{2}bh$ or $A = \frac{bh}{2}$



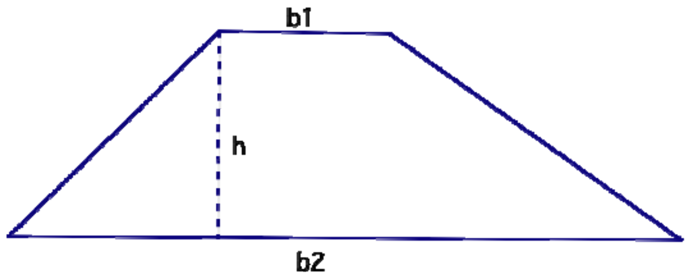
Notice the relationship between the area of a triangle and the area of a parallelogram that have the same base and height. The area of the triangle is one-half the area of the parallelogram.

In Grade 6, students will use *A* = *bh* to derive the **area formula for a trapezoid**.

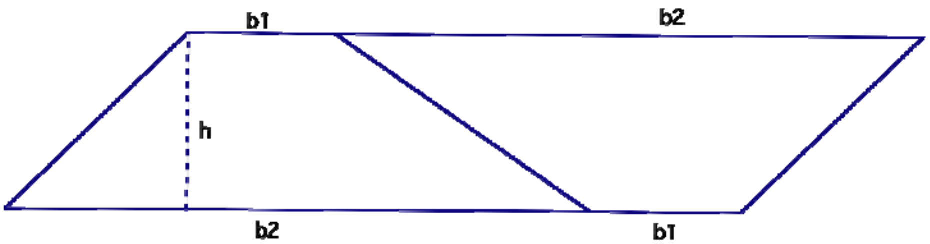
GLE 0606.4.3 Develop and use formulas to determine the circumference and area of circles, and the area of trapezoids, and develop strategies to find the area of composite shapes.

✓ **0606.4.14 Relate the area of a trapezoid to the area of a parallelogram.**

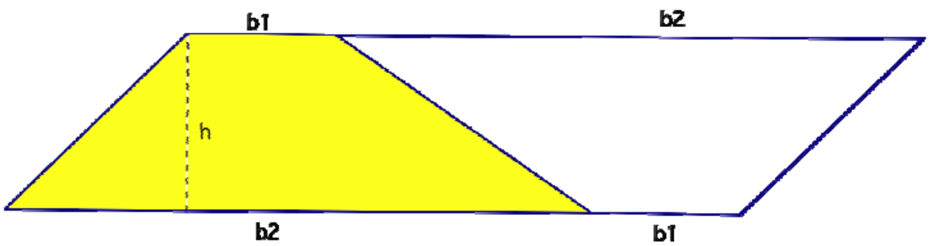
Start with a trapezoid:



Make a copy of it (for example, on patty paper) and use both trapezoids to construct a parallelogram.



Notice the parallelogram has the same height as the original trapezoid, but the base has changed. Using the area formula for a parallelogram, we would say: $A = bh$, where $b = (b_1 + b_2)$ and $h = h$. So the area of this parallelogram is $A = (b_1 + b_2)h$. Notice, however, that the area of the trapezoid is one-half the area of the parallelogram, so the area of the trapezoid is: $A = \frac{1}{2}(b_1 + b_2)h$ or $A =$ $A = \frac{(b_1 + b_2)h}{2}$



Notice the relationship between the area of a trapezoid and the area of a parallelogram that have the same height and related bases. The area of the trapezoid is one-half the area of the parallelogram.

✓0606.4.12 Derive the meaning of Pi using concrete models and/or appropriate technology.

✓0606.4.13 Understand the relationships among the radius, diameter, circumference and area of a circle, and that the ratio of the circumference to the diameter is the same as the ratio of the area to the square of the radius, and that this ratio is called Pi.

To derive Pi, students can measure the circumference and diameter of various-sized circles and calculate the ratio of circumference to diameter. Once this is established, the formula for the circumference of a circle can be derived. Students can then focus on the relationships between the radius, diameter, circumference, and area of the circle.

- ✓ **0606.4.17 Use manipulatives to discover the volume of a pyramid is one-third the volume of the related prism (the heights and base areas are equal).**
- ✓ **0606.4.18 Use manipulatives to discover the volume of a cone is one-third the volume of the related cylinder (the heights and base areas are equal).**

SPI 0606.4.5 Determine the surface area and volume of prisms, pyramids and cylinders.
SPI 0606.4.6 Given the volume of a cone/pyramid, find the volume of the related cylinder/prism or vice versa.

"See-through" geosolids work well for discovering the volume relationships in the above checks. Use rice/sand to pour the contents of one into the other.

For **SPI 0606.4.6**, students use their knowledge of the relationship to solve problems.
For example, given a cone with volume of 10 cubic centimeters, find the volume of the cylinder with the same base and height. Since the relationship between their volumes is such that the volume of a cone is one-third the volume of the related cylinder, the volume of the cylinder is 3 times the volume of the cone. This implies the volume of the cylinder must be 30 cubic centimeters. Parallel problems can be explored for rectangular pyramids and prisms.

<u>Triangle Inequality Property</u>

- ✓ **0606.4.5 Model and use the Triangle Inequality Theorem.**

SPI 0606.4.3 Solve problems using the Triangle Inequality Theorem.

The Triangle Inequality Property states: The sum of the measures of any two sides of a triangle must be greater than the measure of the third side.

Using this property, students are to determine whether three side lengths could form a triangle. Many teachers use straws, spaghetti, or something similar to connect this activity to measurement. Students measure and cut different lengths and see if they can form a triangle. Students should discover the property through investigation.

Content in Focus
Algebra I

Students often do not see connections between the underpinnings of an axiomatic approach and the applications presented to them. That is understandable because most of the information that we receive in life is not initially presented to us in an rule-based format; rather, through experience, we learn what to expect in certain situations, then try to make some rule that explains that phenomenon afterwards.

This has certainly been true for science and even in language arts in the search for grammatical and mechanical rules in communication. But in mathematics, we frequently begin with the axioms and proceed from there, without regard to the fact that these structures often originated in trial-and-error methodologies used to solve real problems. The formal proofs historically succeeded, not preceded, the exploration. Hence, it behooves us, when possible, to at least attempt to introduce our students to the praxis of induction and concrete discovery before we necessarily overlay deduction and abstraction.

And since there are is a lot of material to cover under the new standards, it also behooves us to find ways to compress and collapse what were formerly separate notions, lectures, and lessons into a unity of thought – both for the sake of time, and for the sake of making connections.

Consider a typical unit at the beginning of the year on *properties of the real numbers*. Let a represent any real number ($a \in \Re$): [Note that the symbology is more *caught* than *taught*. As the instructor uses mathematical symbols, notation, and language, the student catches on just as he did when learning to speak his native language as a child.]

Symbolic	English
➢ $a \cdot 1 = a$	Any (real) number times one is itself.
➢ $\frac{a}{a} = 1\ [a \neq 0]$	Any non-zero number divided by itself is one.

Most algebra students would recognize that $5 \cdot 1 = 5$, and that $\frac{5}{5} = 1$

So go ahead and talk about adding fractions now:

$\frac{3}{7} \cdot 1 = \frac{3}{7}$ and since $\frac{5}{5} = 1$, then

$\frac{3}{7} \cdot \frac{5}{5} = \frac{15}{35}$, which is equal to $\frac{3}{7}$

Usually textbooks stop there and review the concept of rational numbers and their operations, which has no doubt been *taught* for several years to the student, but many still only memorize some algorithm for finding the answer rather than understanding *why* it works or developing their own algorithms that make sense.

So, instead of stopping there, go ahead and extrapolate the concept to algebraic fractions, i.e., fractions that contain variables:

$$\frac{x}{7}\cdot 1=\frac{x}{7} \quad \text{and since } \frac{5}{5}=1\text{, then}$$
$$\frac{x}{7}\cdot\frac{5}{5}=\frac{5x}{35}\text{, which is equal to } \frac{x}{7}$$

You see that notion that *algebraic* fractions are somehow different from *regular* fractions (pardon the lack of formal language here, but we are attempting to use terms that students in the early stages of algebra might use in their classroom discussions) is contrived by the textbook. Students see little difference between the two if the instructor makes the transition smoothly.

Indeed, this concept can be quickly carried forward to include *polynomials* once the concept of using the distributive law to include polynomials has been introduced (not necessarily mastered):

$$\frac{(x+2)}{7}\cdot 1=\frac{(x+2)}{7} \quad \text{and since } \frac{5}{5}=1\text{, then}$$

$$\frac{(x+2)}{7}\cdot\frac{5}{5}=\frac{5(x+2)}{35}\text{, which is equal to } \frac{(x+2)}{7}$$

The introduction of the concept of polynomials by using *algebra tiles* can make this a more concrete introduction to an abstract problem, and such an approach is warranted here.

So, the idea that the properties of the real numbers, operations on rational numbers and algebraic fractions (including polynomials) somehow belong to different chapters/units/delineations of time is inefficient, at best, and probably should be considered poor praxis and a recipe for frustration and underachievement, at worst, albeit common.

It is also worthy of note that concepts that are more difficult for students, such as algebraic and polynomial fractions, should be introduced *as early in the school year as possible* to allow *all* students time to master the concepts.

The student can, relatively quickly, be guided through one of the more challenging topics in algebra, *operations with polynomials*, in the first few weeks of the school year, allowing plenty of time for these difficult concepts to *soak in*.

Note how many *Checks for Understanding* can be included in one concept by not artificially separating it into concepts using only constants versus concepts using variables.

Checks for Understanding (Formative/Summative Assessment):

- ✓ 3102.1.1 Develop meaning for mathematical vocabulary.
- ✓ 3102.1.2 Use the terminology of mathematics correctly.
- ✓ 3102.1.3 Understand and use mathematical symbols, notation, and common mathematical abbreviations correctly.
- ✓ 3102.1.9 Identify and use properties of the real numbers (including commutative, associative, distributive, inverse, identity element, closure, reflexive, symmetric, transitive, operation properties of equality).
- ✓ 3102.1.10 Use algebraic properties to develop a valid mathematical argument.
- ✓ 3102.1.11 Use manipulatives to model algebraic concepts.
- ✓ 3102.1.15 Apply arithmetic concepts in algebraic contexts.

Similarly, rather than *tell* students about the effect that coefficients have on graphs, allow them to discover the *rules* for themselves, then *direct* them to ensure that their solutions are mathematically viable with regard to rigor and sufficiency.

For example, in graphing the polynomial $y = ax^2 + bx + c$, rather than explain how the graph changes when the values of a, b, and c are changed, have the students use a graphing utility, such as a graphing calculator or computer application, to explore the concept. Combine this exercise with *completing the square* to form a different method for looking at parabolic graphs: $y = a(x-h)^2 + k$. These are examples of *rich tasks*, from which several checks for understanding, grade level expectations, and state performance indicators may be extracted.

Often in the mathematics classroom, instructors feel obligated to explain a concept thoroughly, then give *word problems* or *real life* examples. Another approach is to give the students a *minimal* instruction set, get them into a lab or exploration, then have them *discover* what the teacher was going to lecture on.

Also, this allows students to jump ahead in the book and be introduced to concepts that are not usually encountered until the later chapters as well as help students make connections among concepts instead of viewing them as separate entities. For example, there is no reason that students cannot graph $y = ax^2 + bx + c$ on the first day of class. All they need is a graphing calculator and minimal instructions on adjusting the *window*.

In data analysis, students enjoy using actual data from their world – surveys of favorite artists among classmates, results from throwing paper wads at a trashcan or dry erase board, etc. There is no need to make an elaborate lecture before doing data analysis. Let the students *discover* what you wanted to tell them by asking them questions about it and having them write about what they learned.

Content in Focus
Geometry

In order to achieve the content and rigor of the new standards, teachers will need to learn how to compress many topics into a few lessons – called *rich tasks* – and extract many objectives from those.

Also, the notion that topics must be introduced in the timeline given by a textbook is generally not ideal. Students are capable of, prefer to, and should combine topics that a typical textbook would consider separately.

The typical geometry teacher follows the textbook. Solids are at the end of the book. But is this the most advantageous way in which to teach the objectives?

What if, early in the school year, students are asked to construct geometric solids, like cylinders, cones, prisms, pyramids, etc.? Students could study two and three-dimensional objects simultaneously. And instead of introducing the formulas for volume first, have students fill their constructed 3-dimensional objects with popcorn or sand or birdseed and measure the volume using graduated cylinders from science class. The Pythagorean theorem, trigonometry, measuring angles, and many more objectives could be covered in just a few constructions. Also, do not think that students need to do that same task everyday until it is mastered. Introduce it, then move to another topic, then re-visit the original topic – only this time, take it deeper. Here are a few of the CLEs and Checks for Understanding that could be covered by such an approach.

CLE 3108.4.1 Develop the structures of geometry, such as lines, angles, planes, and planar figures, and explore their properties and relationships.
CLE 3108.4.2 Describe the properties of regular polygons, including comparative classification of them and special points and segments.
CLE 3108.4.4 Develop geometric intuition and visualization through performing geometric constructions with straightedge/compass and with technology.
CLE 3108.4.5 Extend the study of planar figures to three-dimensions, including the classical solid figures, and develop analysis through cross-sections.
CLE 3108.4.6 Generate formulas for perimeter, area, and volume, including their use, dimensional analysis, and applications.

- ✓ 3108.1.11 Identify and sketch solids formed by revolving two-dimensional figures around lines.
- ✓ 3108.1.1 Check solutions after making reasonable estimates in appropriate units of quantities encountered in contextual situations.
- ✓ 3108.1.2 Determine position using spatial sense with two and three-dimensional coordinate systems.

- ✓ 3108.1.5 Use technology, hands-on activities, and manipulatives to develop the language and the concepts of geometry, including specialized vocabulary (e.g. graphing calculators, interactive geometry software such as Geometer's Sketchpad and Cabri, algebra tiles, pattern blocks, tessellation tiles, MIRAs, mirrors, spinners, geoboards, conic section models, volume demonstration kits, Polydrons, measurement tools, compasses, PentaBlocks, pentominoes, cubes, tangrams).
- ✓ 3108.1.8 Understand how the similarity of right triangles allows the trigonometric functions sine, cosine, and tangent to be defined as ratio of sides.
- ✓ 3108.1.9 Expand analysis of units of measure to include area and volume.
- ✓ 3108.1.10 Use visualization, spatial reasoning, and geometric modeling to solve problems.

In this respect, we can often begin at the end of the textbook and work backward. Most athletes don't learn the rules by reading the rulebook; they learn the rules by playing the game with a referee. Solids can be taught *simultaneously* with two-dimensional geometry. And constructions on a Cartesian plane (graph paper) lend themselves well to analytic geometry.

Another *rich task* that is often overlooked or placed in the category of interesting, but not necessary – especially in a curriculum packed with more material than last year – is that of *tessellations*. Tesselations can provide a forum to discuss almost all of the properties of lines, segments, and polygons – even three-dimensional objects can be studied through tessellations.

The use of construction and study is vital for the geometry teacher who wishes for his students to master the curriculum and enjoy doing so. Just give students straightedge and compass, and give them a project to construct. For example: construct a circle with radius 3.2 cm, and a 30 degree inscribed angle, and a central angle that contains or intersects that same arc.

And for teachers who do not feel creative, there are many books and ancillary materials on this subject, as well as a wealth of expert teachers who have many ideas they would love to share – all they have to do is ask.

Content in Focus
Algebra II

In order to achieve the content and rigor of the new standards, teachers will need to learn how to compress many topics into a few lessons – called *rich tasks* – and extract many objectives from those.

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CLE 3108.4.4 Develop geometric intuition and visualization through performing geometric constructions with straightedge/compass and with technology.

CLE 3108.4.5 Extend the study of planar figures to three-dimensions, including the classical solid figures, and develop analysis through cross-sections.

CLE 3108.4.6 Generate formulas for perimeter, area, and volume, including their use, dimensional analysis, and applications.

- ✓ 3108.1.11 Identify and sketch solids formed by revolving two-dimensional figures around lines.
- ✓ 3108.1.1 Check solutions after making reasonable estimates in appropriate units of quantities encountered in contextual situations.
- ✓ 3108.1.2 Determine position using spatial sense with two and three-dimensional coordinate systems.
- ✓ 3108.1.5 Use technology, hands-on activities, and manipulatives to develop the language and the concepts of geometry, including specialized vocabulary (e.g. graphing calculators, interactive geometry software such as Geometer's Sketchpad and Cabri, algebra tiles, pattern blocks, tessellation tiles, MIRAs, mirrors, spinners, geoboards, conic section models, volume demonstration kits, Polydrons, measurement tools, compasses, PentaBlocks, pentominoes, cubes, tangrams).

✓ 3108.1.8 Understand how the similarity of right triangles allows the trigonometric functions sine, cosine, and tangent to be defined as ratio of sides.
✓ 3108.1.9 Expand analysis of units of measure to include area and volume.
✓ 3108.1.10 Use visualization, spatial reasoning, and geometric modeling to solve problems.

The Algebra II curriculum is full of concepts that are more of an esoteric nature, and as a result, concrete examples are harder to come by for some of these than in Geometry or Algebra I. As in no other math course, Algebra II creates more challenges for students and teachers alike to utilize applications of transcendental and imaginary numbers with which students can relate.

Although *real life* examples for imaginary and complex numbers certainly exist, they are not in the domain of the common experience for most people. Imaginary/complex numbers crop up in electrical engineering, stock market (the Black-Scholes option pricing model), number theory, fluid mechanics, and other places, but they are not everyday occurrences for the rank-and-file. For this reason, students should be encouraged to find models when they exist and are within the scope of this course, but also to study the topics without always needing to find applications – at least immediately.

Much of the Algebra II course should involve students discovering how various functions and their inverses operate. Students should be encouraged to utilize graphing utilities, such as a graphing calculator or computer graphing application, so that many graphs and their transformations can be graphed in a short period of time to allow the student time to see what effect a particular multiplication or addition or exponential or logarithmic operation has on a function. Reflecting a function through the line $y = x$ allows the student to look at a geometric transformation that generates the graph of an inverse.

✓ 3103.3.3 Determine and graph the inverse of a function with and without technology.
✓ 3103.3.4 Analyze the effect of changing various parameters on functions and their graphs.
SPI 3103.3.7 Identify whether a function has an inverse, whether two functions are inverses of each other, and/or explain why their graphs are reflections over the line y = x.

Instead of lecturing on trig functions, have students construct the unit circle, graph radii at 15 degree increments ($\frac{\pi}{12}$ radians), measure the rectangular coordinates on graph paper by hand, record the data as cos A and sin A, respectively, record the values of $\frac{y}{x}$ as tan A, etc., then allow students to graph the angle versus the sin A, cos A, and tan A on separate graphs, along with their reciprocals, csc A, sec A, and cot A.

CLE 3103.4.1 Understand the trigonometric functions and their relationship to the unit circle.
CLE 3103.4.2 Know and use the basic identities of sine, cosine, and tangent as well as their reciprocals.
✓ 3103.1.7 Use the unit circle to determine the exact value of trigonometric functions for commonly used angles (0°, 30°, 45°, 60°...).

This discovery approach works for e^x, *ln(x)*, and other functions that comprise the body of the Algebra II course. Discrete mathematics, with its arithmetic and geometric sequences, are best learned in a lab situation, and with student-led discovery of the various formulas instead of teachers and textbooks presenting them in a lecture or printed format before students have a chance to discover them.

✓ 3103.1.8 Understand and describe the inverse relationship between exponential and logarithmic functions.

✓ 3103.1.9 Translate the syntax of technology to appropriate mathematical notation for non-linear and transcendental functions.

Conic sections can be as concrete as you like. For example, pour flavored gelatin in a pointed ice cream cone, pop it out on a plate, and cut with a fine wire. Students can make their own conic sections before studying them formally. Large parabolic reflectors can be made out of cardboard and aluminum foil and used as solar reflectors to boil water, or as a listening device.

SPI 3103.3.11 Graph conic sections (circles, parabolas, ellipses and hyperbolas) and understand the relationship between the standard form and the key characteristics of the graph.

In other words, allow the students time to explore the concepts *before* they see them in a textbook or the instructor lectures on the topics. This *guided discovery* approach is the best and fastest way for students to gain a feel for the nuances of the functions presented in the Algebra II curriculum.